Working group on prismatic homotopy theory

- **Lecture 1: Prismatic homotopy theory and the work of Carmeli-Feng**
  - Give an overview of Lurie's vision on prismatic homotopy theory.
  - What do Carmeli-Feng do? How are F-gauges used in their story? How do these circle of ideas fit into the prismatic homotopy theory program?
- **Lecture 2, 3: Overview of motivic spectra**
  - Introduce and recall motivic spectra (both $\mathbb{A}^1$ and non-$\mathbb{A}^1$-invariant versions).
  - Go over the construction of syntomic cohomology as a motivic spectrum. For this, might need to briefly review the general procedure for constructing commutative algebra objects out of preoriented graded algebras in MS_k as in [CF25, Sections 3.1-3.6]. Recall in addition, at least over a characteristic zero field K, how to view motivic cohomology from this perspective. (cf [CF25, Section 3.6.2])
  - Mention that the motives corresponding to smooth projective varieties are dualizable with respect to the symmetric monoidal structure on MS_k. This goes by the name of Atiyah duality.
  - Describe the "perfectoid nearby cycles" functor $\psi : \mathrm{SH}_K \to \mathrm{MS}_k$ as in [CF25, Section 4], and how one recovers the identification $\psi((\mathbb{Z}_p^{\mathrm{mot}})_K) \simeq (\mathbb{Z}_p^{\mathrm{syn}})_k$ cf. [CF25, Sections 4.2-4.4]
- **Lecture 4: Relating F-gauges with motivic spectra**
  - Describe the comparison functor $\Upsilon : \mathrm{Mod}_{\mathbb{F}_p^{\mathrm{syn}}}(\mathrm{MS}_k) \to D((\mathrm{Spec} k)^{\mathrm{syn}}|_{\mathbb{F}_p})$ from [CF25, Section 9.3] and its $\mathbb{Z}_p$-linear variant.
  - Describe how [CF25, Theorem 9.1.1], which states that the category of F-gauges over a perfect field k are compactly generated, is used to prove [CF25, Proposition 9.3.7] This proposition asserts that the above comparsion functor induces an equivalence between $D((\mathrm{Spec} k)^{\mathrm{syn}}|_{\mathbb{F}_p})$ and the full subcategory of $\mathrm{Mod}_{\mathbb{F}_p^{\mathrm{syn}}}(\mathrm{MS}_k)$ generated under colimits, twists, and shifts from objects of the form $\Sigma^\infty_+ X \otimes \mathbb{F}_p^{\mathrm{syn}}$, for $X \in \mathrm{Sm}_k$ smooth and projective.
- **Lecture 5: Proof of Theorem 9.1.1**
  - Describe the proof of [CF25, Theorem 9.1.1] in more detail.
- **Lecture 6 (and maybe 7?): Spectral prismatic F-gauges**
  - Begin with constructions of categories of "syntomic spectra" as in [CF25, Section 5]. Include a brief discussion of "motivic Adams approximation" and how the motivic Adams spectral sequence does not converge, in contrast to what happens in stable homotopy theory.
  - Given the equivalence of [CF25, Proposition 9.3.7], give the definition of spectral F-gauges, $\mathrm{FGauge}_{\mathbb{\Delta}}(k)^{\mathrm{pre}}_{\mathbb{S}}$ as it appears in [CF25, Definition 9.2.3]
  - Describe the functor $\mathrm{Sm}_k \to \mathrm{FGauge}_{\mathbb{\Delta}}(k)^{\mathrm{pre}}_{\mathbb{S}}$, c.f. [CF25, Section 9.4] lifting the functor $X \mapsto \mathcal{H}^X$.
  - Discuss spectral Serre duality, as it appears in [CF25, Section 11.3]. This will require a brief recap of Brown-Comenetz duality from stable homotopy theory.
- **Lecture 8: Syntomic Steenrod algebra**
  - Mention the motivating problem , cf [CF25, Section 1.2] and [Voe02] of defining a theory of motivic Steenrod operations "at the defining characteristic".

- Recall the definition of the syntomic Steenrod algebra, cf. [CF25, Definition 6.1.1], and mention the alternative definition in terms of spectral F-gauges, [CF25, Corollary 9.2.7].
- Recall some of Voevodsky's results on the Steenrod algebra in characteristic zero, as in [Voe03].
- The goal is to summarize how perfectoid nearby cycles and Voevodsy's classical computations are used to prove [CF25, Thoerem 1.2.1], which describes the structure of the syntomic Steenrod algebra.
- Describe the "prismatized" version of the syntomic Steenrod algebra.
- Describe how Poincaré duality for syntomic cohomology is compatible with the Steenrod operations; in particular, sketch how [CF25, Theorem 11.1.1] is deduced from the existence of spectral Serre duality. As remarked in the introduction, this compatibility is their main motivation for using F-gauges/prismatization in the first place. (This can also be pushed to the following lecture).

- **Lecture 9, possibly 10: Application to Brauer groups**
  - One or two talks putting this all together to prove the application towards Tate's symplecticity conjecture. More details to come.

References

[CF25] Shachar Carmeli and Tony Feng, *Prismatic steenrod operations and arithmetic duality on brauer groups*, arXiv preprint arXiv:2507.13471 (2025). ↑(document)

[Voe02] Vladimir Voevodsky, *Open problems in the motivic stable homotopy theory. i*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998) **3** (2002), 3–34. ↑(document)

[Voe03] ———, *Reduced power operations in motivic cohomology*, Publications Mathématiques de l'IHÉS **98** (2003), 1–57. ↑(document)